01 The sum of two positive integers is 52 and their LCM is 168. Find the numbers.

Let the positive integers be a and b also $a \leq b$.

Let d = (a, b), so that a = dm, b = dn and (m, n) = 1

Thus

i) a + b = d (m + n) = 4 | 13 and

ii) LCM of a, b = dmn = 168 = 4 × 2 × 7 × 3

But

((m + n)d, mnd) = d [Since (m, n) = 1]

Hence by (i) and (ii) d = 4

So,

m + n = 13 and mn = 42

Hence m = 6, n = 7 and a = dm = 24, b = dn = 28

02 Let S = $1^4 + 2^4 + ... + 2021^4$. Find the remainder when S is divibed by 10.

S = $1^4 + 2^4 + ... + 2021^4$

Number	Units digit
1^4	1
2^4	6
3^4	1
4^4	6
5^4	5
6^4	6
7^4	1
8^4	6
9^4	1
TOTAL	**33**

From 1^4 to 2020, repetition of 1^4 to 9^4 occurs 202 times

Units digit = 33 × 202 = 6666

In 2021^4 = 1

6667

Remainder on division by 10 is 7

03 What are the remainders when a square of any number is divided by 8.

Any number can be assumed

i) 2p, 2p + 1 even or odd

ii) 3p, 3p + 1, 3p + 2 based on the remainder obtained on dividing by 3

iii) 4p, 4p + 1, 4p + 2, 4p + 3 based on the remainders obtained on dividing by 4 and so on

We shall take the form (iii), since it is convenient to divided by 8

Square remainder on dividing by 8

$(4p)^2 = 16p^2$	....	0
$(4p + 1)^2 = 16p^2 + 8p + 1$	....	1
$(4p + 2)^2 = 16p^2 + 16p + 4$	....	4
$(4p + 3)^2 = 16p^2 + 24p + 9$	....	9

Hence square of any number when divided by 8 leaves remainder 0, 1 or 4

04 What are both primes p > 0 for which $\frac{1}{p}$ has a purely periodic demical expansion with a period 5 digits long ? [NOTE: $\frac{1}{37} = 0.\overline{027}$ starts to repeat immediately, so it's purely periodic. Its period is 3 digits long.]

Let's convert a repeating decimal to fractional form, then duplicate the technique to solve this problem.

To express x = $0.\overline{12345}$ as a fraction,

first multiply by 10^5 to get 100 000x = $12345.\overline{12345}$

Subtract to get 99999x = 12345. Finally, x = 12345 / 99999 = 4115 / 33333.

In the problem, $\frac{1}{p} = 0.\overline{abcde}$, so $\frac{10^5}{p} = abcde.\overline{abcde}$.

Subtracting and dividing, $\frac{1}{p} = \frac{abcde}{99999}$. Since p(abcde) = 99999, p is a prime factor of 99999 = 9 × 11111 = 3^2 × 41 × 271. Since $\frac{1}{3} = 0.\overline{3}$ has a period of length 1, the two primes p for which $\frac{1}{p}$ has a period of length 5 are 41,271

[NOTE: $\frac{1}{41} = 0.\overline{02439}$ and $\frac{1}{271} = 0.\overline{00369}$]

05 A theorem due to Fermat (but first proven by Euler) says that "Every prime which is 1 more than a multiple of 4 can be written as the sum of two squares in one and only one way".

For the prime 818101, what are the two positive integers (which in this case, differ by 19) whose squares satisfy this theorm of Fermat ?

Method 1 :

The integers differ by 19. Use this to find a good starting point for trial and error. Assume that x^2 and $(x + 19)^2$ are roughly equal, so each is roughly half of 820 000. Since $\sqrt{410000} \approx 640$, the number 640 is approximately midway between two integers whose difference is 19 and whose squares have a sum of 818 101. The sum $631^2 + 650^2 = 820\ 661$ is too large. Reduce each integer by 1. Since $630^2 + 649^2 = 818\ 101$, the answer is 630,649

Method 2 :

Solve $x^2 + (x + 19)^2 = 818\ 101$, $x > 0$, by the quadratic formula to get $x = 630$, $x + 19 = 649$

(NOTE: Other examples of this theorem are $5 = 2^2 + 1^2$, $13 = 2^2 + 3^2$ and $17 = 4^2 + 1^2$. It's easy to show that 3, 7 and 11 aren't the sum of two squares)